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Double boundary layers in oscillatory viscous flow

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This paper is concerned with unsteady laminar boundary layers on solid bodies in the presence of a fluctuating external flow of small amplitude. Any containing enclosures are assumed to be at infinity. Compressibility is ignored and conditions are given under which this and other approximations are valid. Special attention is focussed on the phenomenon of the formation of a steady-streaming flow, induced by the Reynolds stresses in the oscillatory boundary layer: it is shown that, if the characteristic Reynolds number of the steady streaming is large, there is an outer boundary layer within which the steady-streaming velocity decays to zero. The thickness of this outer layer is large compared with that of the inner (oscillatory) layer, but small compared with a typical dimension of the body.

The partial differential equation for the flow in the outer layer is solved in a typical case by a generalization of a series-expansion method due to Fettis. Similarity solutions of the equation are also described.

The theory is applied specifically to the case of flow generated by a circular cylinder oscillating along a diameter in an infinite fluid. Qualitative agreement is obtained with experiments performed by Schlichting.

1. Introduction

Problems of oscillatory fluid motion often occur in situations where solid boundaries are present. Because of the effect of viscosity, boundary layers are formed at such solid boundaries, and as a result of the unsteadiness they are of a rather complex kind. Some examples of such flows are discussed in this paper, for cases in which the imposed external flow has zero mean.

Compressibility is ignored. The requirements for this approximation have been discussed and stated very clearly by Lighthill (1963, pp. 11–13). Here we quote the two main requirements in isothermal situations: (i) the Mach number must be small compared with unity; and (ii) the wavelength of sound (associated with any given frequency) must be large compared with a typical dimension of the system. If U is a typical speed, d a typical length, ω a typical frequency and c the speed of sound, the requirements may be written

$$\text{(i) } U/c \ll 1 \quad \text{and} \quad \text{(ii) } \omega d/c \ll 1. \tag{1.1}$$

These conditions are met in many experiments involving water waves or oscillating bodies in liquids. Later we shall consider, as a special example, the flow due to a circular cylinder which is oscillating along a diameter. Although this flow is an example of a radiation problem, with sound waves being radiated to infinity, such effects are negligible compared with other effects, provided the wave length of the sound generated (c/ω) is large compared with the cylinder diameter d. Such was the case in experiments to which reference will be made. These and other conditions necessary for the validity of this theory are described at the end of § 2.

The object of this paper is twofold: first, to expose some ambiguities in, and to provide a more satisfactory physical and mathematical model of, the phenomenon of 'steady streaming' in oscillatory flows; and secondly to extend Fettis's (1956) series method of solving some non-linear ordinary differential equations, to the case of certain partial differential equations. In order to provide a focus, attention will be concentrated later on the oscillating-cylinder problem mentioned above; however, the basic physical and mathematical ideas apply in many other situations, for example, in the context of water waves above a solid surface (Longuet-Higgins 1953, 1960). Some of the author's ideas on the subject are summarized elsewhere (Stuart 1963), while related ideas for the torsionally oscillating disk problem have been discussed by Rosenblat (1959) and, more recently, by Benney (1964); the latter problem is discussed in Appendix 2.

2. General analysis

Let us consider a two-dimensional flow bounded by a solid surface, with co-ordinates fixed in the surface; as usual in laminar-boundary-layer theory, surface curvature will be neglected, but this approximation receives some attention at the end of this section. Let x denote the co-ordinate parallel to the wall, z the co-ordinate normal to it, and u, w the corresponding velocity components. In addition, let t denote time, p pressure, ρ density, ν kinematic viscosity, and $U(x,t)$ the external-flow velocity.

Defining a stream function ψ by

$$u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x, \tag{2.1}$$

we may write the boundary-layer equation in the form

$$\frac{\partial^2\psi}{\partial z\,\partial t} + \frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial x\,\partial z} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial z^2} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^3\psi}{\partial z^3}. \tag{2.2}$$

We wish to confine our attention to functions $U(x,t)$ of the form

$$U(x,t) = \tfrac{1}{2}U_0(x)\,(e^{i\omega t} + e^{-i\omega t}). \tag{2.3}$$

It can be argued (cf. Schlichting 1932) that the oscillatory boundary layer on the wall will have a thickness of order $(\nu/\omega)^{\frac{1}{2}}$ owing to the diffusion of vorticity with period $2\pi/\omega$. If we now write

$$\left.\begin{aligned} U(x,t) = U_\infty\, V(\xi,\tau), \quad x = \xi d, \quad z = \eta(2\nu/\omega)^{\frac{1}{2}}, \\ \psi = (2\nu/\omega)^{\frac{1}{2}}\, U_\infty\, \chi(\xi,\eta,\tau), \quad t = \tau/\omega, \quad U_0(x) = U_\infty V_0(\xi), \quad \alpha = U_\infty/\omega d \end{aligned}\right\}, \tag{2.4}$$

where d is a characteristic length and U_∞ a characteristic speed, equation (2.2) becomes

$$\frac{\partial^2\chi}{\partial\eta\,\partial\tau} - \frac{1}{2}\frac{\partial^3\chi}{\partial\eta^3} - \frac{\partial V}{\partial\tau} = \alpha\left(-\frac{\partial\chi}{\partial\eta}\frac{\partial^2\chi}{\partial\xi\,\partial\eta} + \frac{\partial\chi}{\partial\xi}\frac{\partial^2\chi}{\partial\eta^2} + V\frac{\partial V}{\partial\xi}\right). \tag{2.5}$$

A solution may be developed in powers of α, for small values of this parameter. The implications of this are considered at the end of this section.

Bearing in mind that

$$V(\xi,\tau) = \tfrac{1}{2}V_0(\xi)\,(e^{i\tau}+e^{-i\tau}), \tag{2.6}$$

we may develop a solution for χ in the form

$$\chi = \tfrac{1}{2}V_0(\xi)\,[\chi_0(\eta)\,e^{i\tau}+\tilde{\chi}_0(\eta)\,e^{-i\tau}]+\alpha[\chi_s(\xi,\eta)+\tfrac{1}{2}(\chi_2\,e^{2i\tau}+\tilde{\chi}_2\,e^{-2i\tau})]+O(\alpha^2). \tag{2.7}$$

The boundary conditions that we wish to impose are

$$\left.\begin{aligned} \chi_0 = d\chi_0/d\eta = \chi_s = \partial\chi_s/\partial\eta = \chi_2 = \partial\chi_2/\partial\eta = 0 \quad \text{at} \quad \eta = 0,\\ d\chi_0/d\eta \to 1, \quad \partial\chi_s/\partial\eta \to 0, \quad \partial\chi_2/\partial\eta \to 0 \quad \text{as} \quad \eta \to \infty, \end{aligned}\right\} \tag{2.8}$$

with appropriate initial conditions. Substitution of (2.7) into (2.5) and solution of the equation for the terms independent of α yields

$$\partial\chi_0/\partial\eta = 1-e^{-(1+i)\eta}, \quad \chi_0 = -\tfrac{1}{2}(1-i)\,[1-e^{-(1+i)\eta}]+\eta. \tag{2.9}$$

The terms of order α obtained from (2.5) and (2.7) yield equations for χ_s and χ_2. For the latter the boundary conditions (2.8) can be satisfied and no difficulty arises. However, in the case of χ_s, it is found to be impossible to satisfy the condition $\partial\chi_s/\partial\eta \to 0$ as $\eta \to \infty$, if the usual conditions of zero velocity are imposed at the wall. The reason for this can be seen, as follows, from the differential equation for χ_s:

$$\frac{\partial^3\chi_s}{\partial\eta^3}+V_0\frac{dV_0}{\partial\xi}\left\{1-\frac{\partial\chi_0}{\partial\eta}\frac{\partial\tilde{\chi}_0}{\partial\eta}+\frac{1}{2}\left(\chi_0\frac{\partial^2\tilde{\chi}_0}{\partial\eta^2}+\tilde{\chi}_0\frac{\partial^2\chi_0}{\partial\eta^2}\right)\right\} = 0. \tag{2.10}$$

Since the complementary function is $(A+B\eta+C\eta^2)$, where A, B and C are functions of ξ, it would be necessary to put two arbitrary functions, B and C, equal to zero if the conditions $\partial\chi_s/\partial\eta \to 0$, $\eta \to \infty$, were to be satisfied; it would then not be possible to satisfy both the boundary conditions at the wall. Consequently, within the framework of this theory the condition at infinity is relaxed to '$\partial\chi_s/\partial\eta$ remains finite when $\eta \to \infty$'; this implies $C = 0$.

The solution for χ_s with this altered boundary condition is found to be

$$\left.\begin{aligned} \chi_s' &= V_0\frac{dV_0}{d\xi}\{-\tfrac{3}{4}+\tfrac{1}{4}e^{-2\eta}+2e^{-\eta}\sin\eta+\tfrac{1}{2}e^{-\eta}\cos\eta-\tfrac{1}{2}\eta e^{-\eta}(\cos\eta-\sin\eta)\},\\ \chi_s &= V_0\frac{dV_0}{d\xi}\{\tfrac{13}{8}-\tfrac{3}{4}\eta-\tfrac{1}{8}e^{-2\eta}-\tfrac{3}{2}e^{-\eta}\cos\eta-e^{-\eta}\sin\eta-\tfrac{1}{2}\eta\,e^{-\eta}\sin\eta\}. \end{aligned}\right\} \tag{2.11}$$

When $\eta \to \infty$ the steady velocity components take the forms

$$u_s \sim -\frac{3}{4\omega}U_0\frac{dU_0}{dx}, \tag{2.12}$$

$$w_s \sim \frac{3}{4\omega}\frac{d}{dx}\left(U_0\frac{dU_0}{dx}\right)\left\{z-\frac{13}{6}\left(\frac{2\nu}{\omega}\right)^{\frac{1}{2}}\right\}. \tag{2.13}$$

This steady motion is produced by the Reynolds stresses associated with the oscillatory viscous flow.

Having obtained (2.12), which shows that $u_s \nrightarrow 0$ at the edge of the boundary layer, which we shall now call the 'inner layer', we now ask if there are some omitted (non-linear) terms which, through their omission, have brought about this result. A dimensional analysis is now undertaken as an initial step in our understanding of the difficulty. From (2.12) a characteristic velocity is $U_\infty^2/\omega d$; then with a characteristic length d, we can form the Reynolds number

$$R_s = U_\infty^2/\omega\nu, \tag{2.14}$$

and this parameter is clearly of importance in determining just how u_s decays to zero with distance from the wall. Schlichting (1932) argued that, for small values of R_s, the flow outside the inner layer is governed by the linearized Navier–Stokes equations of slow motion; the boundary conditions on u and w at the wall are that $u = u_s$ (2.12) and that $w_s = 0$ (although, as (2.13) implies, it is non-zero but of very small order, $(\nu/\omega d^2)^{\frac{1}{2}}$ times u_s). The boundary condition at infinity is that u shall tend to zero.

The assumption of small values of R_s is implicit in Rayleigh's (1883) paper, and his work and Schlichting's have been discussed by Stuart (1963). With reference to his own experiments, which entailed R_s of order 300, Schlichting mentioned the deficiency of the theoretical approach valid for small values of R_s; but he did not treat large values. On the other hand, Longuet-Higgins (1953, pp. 546, 547, 565) suggested that, for large values of R_s, the steady flow outside the inner layer can be calculated by the theory of inviscid rotational flow; but that suggestion seems physically inconsistent with the boundary conditions (tangential velocity, $u = u_s$, given at the outer edge of the inner layer). Longuet-Higgins did mention, however, that an inviscid rotational solution would not always be possible; moreover (1960) he discussed the relevance of a diffusive layer, of thickness $(\nu t)^{\frac{1}{2}}$, for the non-oscillatory flow.

Here we argue that, when R_s is large, there is a second boundary layer (the 'outer' layer) within which the steady flow, which is 'driven' by the velocity (2.12), tends to zero. Since a typical velocity is $U_\infty^2/\omega d$, and a typical length is d, the thickness of the outer (laminar) layer must have magnitude $d(\omega\nu)^{\frac{1}{2}}/U_\infty$. This is much larger than the thickness $(\nu/\omega)^{\frac{1}{2}}$, of the inner layer, because $\alpha \ll 1$.

Let us consider the mechanics of the flow in the outer layer on the assumption (which is discussed in Appendix 1 and has been justified by Riley (1965) to the present approximation) that the steady flow does not interact with the co-existent oscillatory potential flow. We define

$$x = \xi d, \quad z = \zeta d(\omega\nu)^{\frac{1}{2}}/U_\infty, \quad \psi_s = U_\infty(\nu/\omega)^{\frac{1}{2}}\,\phi(\xi,\zeta), \tag{2.15}$$

and substitute in (A 6) of Appendix 1 to obtain

$$\frac{\partial\phi}{\partial\zeta}\frac{\partial^2\phi}{\partial\xi\,\partial\zeta} - \frac{\partial\phi}{\partial\xi}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{\partial^3\phi}{\partial\zeta^3}, \tag{2.16}$$

with boundary conditions

$$\partial\phi/\partial\zeta = X(\xi), \quad \phi = 0 \quad \text{at} \quad \zeta = 0; \quad \partial\phi/\partial\zeta \to 0 \quad \text{as} \quad \zeta \to \infty, \tag{2.17}$$

where

$$X(\xi) = -\tfrac{3}{4}V_0(dV_0/d\xi). \tag{2.18}$$

An appropriate initial condition will be introduced as required.

We now discuss (2.17) in the light of the condition $\alpha \ll 1$. Since the outer layer has thickness $O(\alpha^{-1})$ times that of the inner layer, the first condition (2.17), which matches the steady tangential velocities in the inner and outer layers, may be applied at $\zeta = 0$. Moreover, since, from (2.13) and (2.15), w in the outer layer is typically $O(\alpha^{-1})$ times w in the inner layer, we may set the former equal to zero at $\zeta = 0$; this yields the second condition (2.17).

It will be recognized that, in the discussion so far given, we have derived what may be regarded as the dominant equations in an 'inner-outer' expansion scheme. In an analysis of that type we would not automatically equate to zero the coefficient C of the complementary function of (2.10); this was emphasized to the author by Dr N. Riley, who has shown that, to the order of the present work, the constant C *must* be zero (Riley 1965). In justification of the present, more pragmatic approach, we suggest that, since the neglected terms of (2.10) are important only towards the edge of the inner layer, little effect of those terms can be expected near the wall. Consequently we expect Schlichting's solution (2.11) to be valid near the wall with, as a corollary, $C = O(\alpha^p)$, $p > 0$; thus we set $C = 0$ for our present purposes, which is that of obtaining the main effects for $\alpha \ll 1$.

We note, however, that if we wished to account for high-order terms in the expansion (2.7), the present arguments would not be valid: the method of inner and outer expansions (as in Riley's work) or the method of two scales, as in the work of Benney (1964), might then be used.

It may be helpful to note that we can consider a single *model* equation which includes both inner and outer layers for χ_s, namely

$$\frac{\partial^3\chi_s}{\partial\eta^3}+V_0\frac{dV_0}{d\xi}\left\{1-\frac{\partial\chi_0}{\partial\eta}\frac{\partial\tilde{\chi}_0}{\partial\eta}+\frac{1}{2}\left(\chi_0\frac{\partial^2\tilde{\chi}_0}{\partial\eta^2}+\tilde{\chi}_0\frac{\partial^2\chi_0}{\partial\eta^2}\right)\right\} = 2\alpha^2\left(\frac{\partial\chi_s}{\partial\eta}\frac{\partial^2\chi_s}{\partial\xi\,\partial\eta}-\frac{\partial\chi_s}{\partial\xi}\frac{\partial^2\chi_s}{\partial\eta^2}\right), \tag{2.19}$$

with conditions

$$\partial\chi_s/\partial\eta = \chi_s = 0 \quad (\eta=0); \quad \partial\chi_s/\partial\eta\to 0 \quad (\eta\to\infty). \tag{2.20}$$

If we let $\alpha \to 0$ with η and χ_s fixed, we obtain (2.10), but this procedure is not uniformly valid. In order to obtain (2.16) for the outer layer we first define

$$\chi_s = 2^{-\frac{1}{2}}\alpha^{-1}\phi(\xi,\zeta), \quad \eta = 2^{-\frac{1}{2}}\alpha^{-1}\zeta, \tag{2.21}$$

and then let $\alpha \to 0$ with ϕ, ζ fixed. The terms proportional to $V_0\,dV_0/d\xi$ disappear because, compared with the other terms as a ratio, they are of a form, namely

$$\alpha^{-3}\exp\left(-k\alpha^{-1}\zeta\right),$$

which tends to zero as $\alpha \to 0$. We may expect the inner and outer layers to merge in the way shown in figure 1, with the maximum value of u_s at the edge of the inner layer.

The transformation (2.21) with $\alpha \to 0$, as applied to (2.19), indicates in part why the applied pressure field, which is included in the $V_0\,dV_0/d\xi$ terms of (2.19), does not affect the equation for the outer layer. Within the framework of customary boundary-layer theory, no modification of the pressure field is possible; pressure changes can be calculated only by considering interactions with the external (inviscid) flow.

The conditions under which we may expect validity of the analysis, which leads to (2.16) and (2.17), are the following:

(i) $U_\infty/c \ll 1$ (small Mach number).

(ii) $\omega d/c \ll 1$ (sound waves unimportant).

(iii) $\alpha = U_\infty/\omega d \ll 1$ (small amplitude of oscillation, so that (2.7) converges quickly).

(iv) $\beta = \nu/\omega d^2 \ll 1$ (boundary-layer approximation for inner layer).

(v) $R_s = U_\infty^2/\omega\nu \gg 1$ (boundary-layer approximation for outer layer).

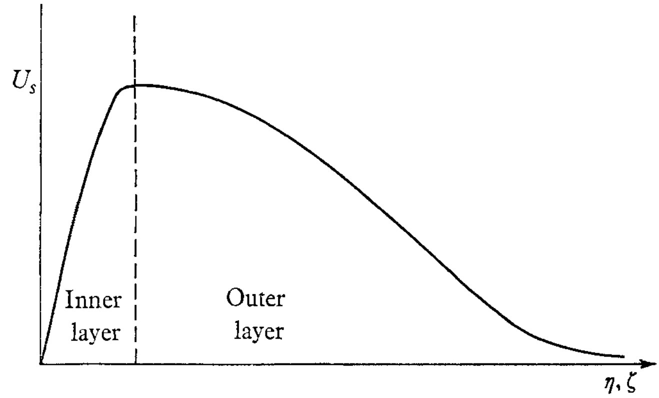

FIGURE 1. Inner and outer layers.

(vi) Other solid or free boundaries are supposed to be remote from the region where the boundary layer is being considered, so that appropriate boundary conditions may be supposed applied at infinity. This condition implies that any steady streaming motion developed is unaffected by other boundary regions (for example, by a large diameter container).

(vii) We have hypothesized that the steady flow within the outer layer is not significantly affected by the simultaneous presence of the potential flow; the meaning of this hypothesis is made explicit in Appendix 1, where the justification given by Riley (1965), who used an 'inner-outer' expansion, is discussed.

(viii) Attention has been restricted to laminar flow. This presumably requires some restrictions on the above parameters, but in the light of present knowledge it is difficult to be very precise. We note, however, that the *local* Reynolds number is likely to be important; for both the inner and outer layers this number, based on local reference velocities and boundary-layer thicknesses, is $R_s^{\frac{1}{2}}$. Suitably small values of this parameter would probably be conducive to laminar flow (from knowledge of results of stability theory of steady flows, one anticipates that this criterion can be consistent with that of R_s large).

In view of the fact that we have used boundary-layer theory (2.2), some discussion of this theory is desirable with reference to conditions (iii), (iv) and (v). If there is no surface curvature the (viscous) terms neglected in (2.2) are of order β; but if surface curvature of order d is present the neglected (viscous) terms are of order $\beta^{\frac{1}{2}}$ times those retained. The latter case is the more usual; moreover since

$R_s = \alpha^2/\beta$, it can be seen that, if R_s is very large, α must be large compared with $\beta^{\frac{1}{2}}$. Consequently, if R_s is large, we may feel justified in neglecting terms $O(\beta^{\frac{1}{2}})$ and in using boundary-layer theory, followed by expansion of the solution of (2.2) in powers of α—at least to some power of α dependent on the size of R_s.

If, on the other hand, R_s is $O(1)$ can we justify Schlichting's use of boundary-layer theory for the inner layer, followed by slow-motion theory for the steady flow in the outer region? In such cases the neglected term $O(\beta^{\frac{1}{2}})$ is comparable in magnitude with the $O(\alpha)$ terms of the expansion (2.7) of (2.2). Such a procedure can, however, be justified as follows, at least to $O(\alpha)$ in (2.7), this being the order to which Schlichting took the expansion. The point is that the $O(\beta^{\frac{1}{2}})$ and $O(\alpha)$ terms represent different physical effects: the first gives a viscous correction to the fundamental oscillation (frequency ω), while the second represents non-linear generation of both a harmonic (2ω) and of a steady-streaming motion. If we calculate the $O(\beta^{\frac{1}{2}})$ correction to the fundamental, equation (2.10) indicates that a steady-streaming correction $O(\alpha\beta^{\frac{1}{2}})$ will follow. The latter is small compared with the $O(\alpha)$ steady streaming. But, if it is desired to calculate *higher-order terms* when R_s is $O(1)$, caution is required, because the $O(\alpha\beta^{\frac{1}{2}})$ term may be as important as the term $O(\alpha^2)$; if this is the case, equation (2.2) is not valid at this order. We emphasize, however, that this problem does not arise when R_s is large, as in this paper.

A comment on boundary-layer separation is also desirable; condition (iii) ensures that this will not arise. We can justify this by reference to the case of a circular cylinder moving with speed $U_\infty \cos \omega t$ along a diameter d; then the potential flow relative to the body is of the form (2·3). If the body were accelerated *uniformly* from rest (Watson 1955; Stuart 1963, p. 374), separation would occur only after the body had traversed a distance $O(0{\cdot}3d)$; on the other hand, for the actual motion the body is accelerating (albeit non-uniformly) from rest for a distance U_∞/ω. If U_∞/ω is very small compared with $0{\cdot}3d$ the flow cannot allow separation. This condition is equivalent to (iii).

3. Solution for the outer layer

In this section we shall solve equation (2.16) in series, subject to the boundary conditions (2.17). The method we shall adopt is one which has been devised by Fettis (1956) for the solution of the ordinary differential equations for rotating-disk and rotating-fluid flows. Here it is shown that a similar method can be used for the partial differential equation (2.16). The method may be regarded either as a formal expansion scheme, in which a 'small' parameter ϵ is first introduced and then (hopefully, for practical convergence) set equal to unity, or as an iterative scheme (as emphasized by Watson 1965). It is also worthy of mention that Fettis's method is remarkably easy to apply when the boundary layer accurately has a behaviour like e^{-kz}, where k is a constant or a function of x, as $z \to \infty$, for then the analysis involves only elementary operations on exponential functions of this kind.

We rewrite the boundary conditions as

$$\partial\phi/\partial\zeta = \epsilon X(\xi), \quad \phi = 0 \ (\zeta = 0); \quad \partial\phi/\partial\zeta \to 0 \quad (\zeta \to \infty), \tag{3.1}$$

and expand the solution of (2.16) in the form

$$\phi = \gamma(\xi) + \epsilon\phi_1(\xi,\zeta) + \epsilon^2\phi_2(\xi,\zeta) + \dots. \tag{3.2}$$

The quantity ϵ is a parameter, which later we shall set equal to unity; $\gamma(\xi)$ is a function of ξ to be determined and gives the form of ϕ as ζ tends to infinity. In the following analysis primes denote derivatives with respect to ζ in the ϕ functions, derivatives with respect to ξ being denoted by suffixes. Derivatives of $\gamma(\xi)$ are denoted by primes. The boundary conditions on the functions ϕ_1, ϕ_2, etc., are

$$\left.\begin{aligned} \phi_1' = X(\xi), \quad \phi_n' = 0 \quad (n = 2, 3, 4, \dots, \zeta = 0), \\ \phi_m, \quad \phi_m' \to 0 \quad (m = 1, 2, 3, 4, \dots, \zeta \to \infty). \end{aligned}\right\} \tag{3.3}$$

The condition $\phi = 0$, $\zeta = 0$, is applied finally and determines $\gamma(\xi)$. It can be seen that, if the series (3.2) converges (in practice) as far as $\epsilon = 1$, we have a valid solution to our problem.

Substituting (3.2) in (2.16) we obtain the following from the terms linear in ϵ

$$\phi_1''' + \gamma'\phi_1'' = 0. \tag{3.4}$$

The solution subject to the boundary conditions is

$$\phi_1' = X\,e^{-\gamma'\zeta}, \quad \phi_1 = -(X/\gamma')\,e^{-\gamma'\zeta}. \tag{3.5}$$

A comment in favour of the Fettis method is noted here: from (3.2), (3.4), (3.5) and later formulae we see that it gives the correct exponential behaviour at infinity, whereas many other series methods do not.

The equations arising from the terms proportional to ϵ^2, ϵ^3, ϵ^4, ϵ^5 are

$$\phi_2''' + \gamma'\phi_2'' = (\gamma''/\gamma')\,X^2 e^{-2\gamma'\zeta}, \tag{3.6}$$

$$\phi_3''' + \gamma'\phi_3'' = -(\gamma''^2/\gamma'^4)\,X^3 e^{-2\gamma'\zeta} + [(\gamma'''/4\gamma'^3) + (\gamma''^2/2\gamma'^4)]\,X^3 e^{-3\gamma'\zeta}, \tag{3.7}$$

$$\phi_4''' + \gamma'\phi_4'' = l_2 e^{-2\gamma'\zeta} + l_3 e^{-3\gamma'\zeta} + l_4 e^{-4\gamma'\zeta}, \tag{3.8}$$

$$\phi_5''' + \gamma'\phi_5'' = m_2 e^{-2\gamma'\zeta} + m_3 e^{-3\gamma'\zeta} + m_4 e^{-4\gamma'\zeta} + m_5 e^{-5\gamma'\zeta}, \tag{3.9}$$

where

$$\left.\begin{aligned} l_2 &= \tfrac{5}{4}(\gamma''^3 X^4/\gamma'^7) - (\gamma'' X^4/12\gamma'^7)\,(\gamma'\gamma''' + 2\gamma''^2), \\ l_3 &= -\tfrac{3}{8}(\gamma'' X^4/\gamma'^7)\,(\gamma'\gamma''' + 2\gamma''^2), \\ l_4 &= (X^4/36\gamma'^7)\,(4\gamma'\gamma''\gamma''' + \gamma^{\mathrm{iv}}\gamma'^2 + 7\gamma''^3), \end{aligned}\right\} \tag{3.10}$$

$$\left.\begin{aligned} m_2 &= (-X\gamma''/\gamma'^3)\,(\tfrac{7}{5}l_2 + \tfrac{16}{45}l_3 + \tfrac{1}{6}l_4), \\ m_3 &= \frac{X^5}{\gamma'^8}\left(\frac{\gamma''^4}{\gamma'^2} + \frac{7}{16}\frac{\gamma''^2\gamma'''}{\gamma'} - \frac{1}{32}\gamma'''^2\right), \\ m_4 &= \frac{X^5}{\gamma'^8}\left(-\frac{7}{18}\frac{\gamma''^4}{\gamma'^2} - \frac{2}{9}\frac{\gamma''^2\gamma'''}{\gamma'} - \frac{1}{18}\gamma^{\mathrm{iv}}\gamma''\right), \\ m_5 &= \frac{X^5}{144\gamma'^8}\left(\frac{25}{6}\frac{\gamma''^4}{\gamma'^2} + \frac{119}{12}\frac{\gamma''^2\gamma'''}{\gamma'} - \frac{\gamma'''^2}{2} + \frac{5}{3}\gamma^{\mathrm{iv}}\gamma'' + \tfrac{1}{4}\gamma^{\mathrm{v}}\gamma'\right). \end{aligned}\right\} \tag{3.11}$$

The solutions of these equations, subject to the boundary conditions, are

$$\phi_2 = (\gamma''/2\gamma'^4)\, X^2(e^{-\gamma'\zeta} - \tfrac{1}{2}e^{-2\gamma'\zeta}), \tag{3.12}$$

$$\begin{aligned}\phi_3 = &-(\gamma''^2/2\gamma'^2)\, X^3(e^{-\gamma'\zeta} - \tfrac{1}{2}e^{-2\gamma'\zeta}) \\ &+ (X^3/24\gamma'^7)\,(\gamma'\gamma''' + 2\gamma''^2)\,(e^{-\gamma'\zeta} - \tfrac{1}{3}e^{-3\gamma'\zeta}),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\phi_4 = &(l_2/2\gamma'^3)\,(e^{-\gamma'\zeta} - \tfrac{1}{2}e^{-2\gamma'\zeta}) + (l_3/6\gamma'^3)\,(e^{-\gamma'\zeta} - \tfrac{1}{3}e^{-3\gamma'\zeta}) \\ &+ (l_4/12\gamma'^3)\,(e^{-\gamma'\zeta} - \tfrac{1}{4}e^{-4\gamma'\zeta}),\end{aligned} \tag{3.14}$$

$$\begin{aligned}\phi_5 = &(m_2/2\gamma'^3)\,(e^{-\gamma'\zeta} - \tfrac{1}{2}e^{-2\gamma'\zeta}) + (m_3/6\gamma'^3)\,(e^{-\gamma'\zeta} - \tfrac{1}{3}e^{-3\gamma'\zeta}) \\ &+ (m_4/12\gamma'^3)\,(e^{-\gamma'\zeta} - \tfrac{1}{4}e^{-4\gamma'\zeta}) + (m_5/20\gamma'^3)\,(e^{-\gamma'\zeta} - \tfrac{1}{5}e^{-5\gamma'\zeta}).\end{aligned} \tag{3.15}$$

Higher-order terms may be calculated in a similar way.

If now we set $\epsilon = 1$ and satisfy the boundary condition $\phi = 0$ at $\zeta = 0$, we obtain the following ordinary differential equation

$$\begin{aligned}\tfrac{1}{2}(\gamma^2)' - X + \frac{\gamma'' X^2}{4\gamma'^3} + \frac{X^3}{36\gamma'^6}(\gamma'\gamma''' - 7\gamma''^2) \\ + (\tfrac{1}{4}l_2 + \tfrac{1}{9}l_3 + \tfrac{1}{16}l_4)/\gamma'^2 + (\tfrac{1}{4}m_2 + \tfrac{1}{9}m_3 + \tfrac{1}{16}m_4 + \tfrac{1}{25}m_5)/\gamma'^2 + \ldots = 0.\end{aligned} \tag{3.16}$$

This formula represents the condition that the entrainment into the outer layer equals the mass flow in the (x)-direction. It is hoped that, for a given function X, a solution for γ can be obtained to good accuracy by including only a few terms in the series. We shall not discuss boundary conditions on (3.16), since we are concerned below only with a special series solution.

In general we cannot expect to solve (3.16) in closed form. Let us consider physical situations with a degree of symmetry such that $X(\xi)$ can be expanded in the form

$$X(\xi) = a^2\xi(1 + \beta_2\xi^2 + \beta_4\xi^4 + \ldots) \tag{3.17}$$

near $\xi = 0$, where a is supposed to be positive. On substitution in (3.16) we find a solution for γ, subject to $\gamma(0) = 0$, as

$$\gamma(\xi) = a\xi\left\{1 + \tfrac{3}{17}\beta_2\xi^2 + \left(\frac{6{,}201}{1{,}116{,}985}\beta_2^2 + \frac{60}{773}\beta_4\right)\xi^4 + \ldots\right\}. \tag{3.18}$$

Especially we note that, for $X(\xi) = a^2\xi$ exactly, (3.5), (3.17) and (3.18) yield

$$\phi = a\xi(1 - e^{-a\zeta}), \tag{3.19}$$

which is an exact solution of (2.16) and (2.17). (The higher-order terms, (3.12) to (3.15), are zero in this case.)

Solutions for more general power expansions than (3.17) could be calculated, but we shall consider only forms of (3.17) in this paper.

Finally, we note that 'similarity' solutions of (2.16), in which it reduces to an ordinary differential equation, are available and are described in Appendix 2.

4. The outer layer on an oscillating circular cylinder

The prototype problem studied by Schlichting (1932) was that of a circular cylinder oscillating along a diameter in a fluid at rest. In this case the characteristic length d is the diameter of the cylinder, and we have

$$U_0(x) = 2U_\infty \sin 2\xi = U_\infty V_0(\xi), \tag{4.1}$$

when axes are fixed in the cylinder. From (2.18) we obtain

$$X(\xi) = -3\sin 4\xi. \tag{4.2}$$

With $\xi = 0, \frac{1}{2}\pi$ denoting the extremities of the diameter along which the cylinder oscillates (figure 2), we shall be concerned with the region $\frac{1}{4}\pi \leqslant \xi < \frac{1}{2}\pi$. The steady flow is symmetrical about $\xi = \frac{1}{4}\pi$.

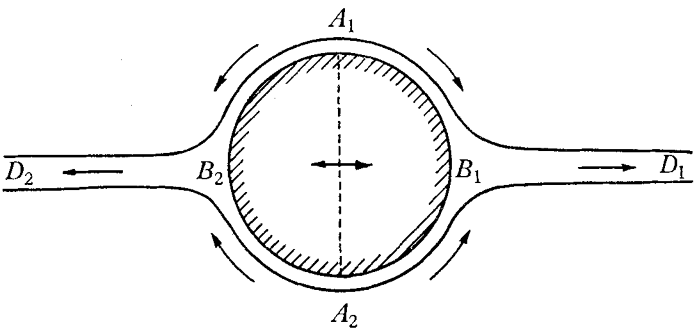

FIGURE 2. Outer layer and steady flow due to an oscillating circular cylinder.

Expanding $X(\xi)$ in the form (3.17) but with ξ replaced by ξ_1, namely

$$X(\xi) = 12\xi_1 - 32\xi_1^3 + \tfrac{128}{5}\xi_1^5 - \ldots, \tag{4.3}$$

where

$$\xi_1 \equiv \xi - \tfrac{1}{4}\pi, \tag{4.4}$$

we obtain the following from (3.18)

$$\gamma(\xi) = 2\sqrt{3}\left(\xi_1 - \tfrac{8}{17}\xi_1^3 + \frac{229{,}056}{1{,}116{,}985}\xi_1^5 + \ldots\right), \tag{4.5}$$

together with

$$\gamma' = 2\sqrt{3}\left(1 - \tfrac{24}{17}\xi_1^2 + \frac{229{,}056}{223{,}397}\xi_1^4 + \ldots\right). \tag{4.6}$$

The stream function ϕ is given by (3.2) with $\epsilon = 1$, (3.5), (3.10) to (3.15), (4.5) and (4.6), and is valid to order ξ_1^5.

The displacement thickness of the outer layer is defined by

$$\delta_{10} = \frac{d}{U_\infty}\frac{(\nu\omega)^{\frac{1}{2}}}{X(\xi)}\int_0^\infty \frac{\partial\phi}{\partial\zeta}\,d\zeta = \frac{\gamma(\xi)}{X(\xi)}\frac{d}{U_\infty}(\omega)^{\frac{1}{2}}\nu. \tag{4.7}$$

In the case of the oscillating circular cylinder this leads to

$$\delta_{10} = \frac{d}{U_\infty}\left(\frac{\nu\omega}{12}\right)^{\frac{1}{2}}[1 + \tfrac{112}{51}\xi_1^2 + \sigma\xi_1^4 + \ldots], \tag{4.8}$$

where $\sigma = 3{\cdot}92794$ approximately. (Formula (4.8) agrees exactly, to the significant figures given, with an independent calculation, due to Riley (1965), by another method.) The solution described above shows that (figure 2) fluid is dragged steadily in the directions A_1 to B_1 and B_2 and A_2 to B_1 and B_2. Fluid moves in radially to balance this. However, it seems unlikely that the series solution converges as far as B_1 and $B_2(\xi_1 = \pm\frac{1}{4}\pi)$. Equation (4.6) suggests $\xi_1 = \frac{1}{2}$ as a possible practical limit of convergence.

At the points B_1 and B_2, the outer boundary layers from the two sides impact, and the details of the solution in this region are unknown. [A similar feature arises in the case of flow due to a rotating sphere, and for discussions of the impacting boundary layers in this case the reader is referred to Howarth (1951) and to Stewartson (1958).] After impact the outer-layer fluid moves in the direction $B_1 D_1$ and $B_2 D_2$; at large distances from the cylinder, and in the absence of other boundaries nearby it is plausible to conjecture that the flow tends to the two-dimensional (Bickley) jet solution of the Navier–Stokes equation, since at such distances the main effect of the cylinder's movement is to provide a source of momentum for the flow. (This suggestion is somewhat analogous to Stewartson's, namely, that in the rotating-sphere case the flow takes on the form of a radial jet far from the equator of the sphere.) Outside of the outer layer, and of the consequent jet, the steady flow is expected to be relatively slight.

Experiments on the flow due to an oscillating circular cylinder have been carried out by Andrade (1931) in air and by Schlichting (1932) in water. In the former experiments R_s was of order 0·5 to 1·0, whereas it was of order 250 in those of Schlichting; for this reason we concentrate on the latter experiments.

With a cylinder diameter of 8 cm, $\omega = 3{\cdot}1\,\mathrm{sec}^{-1}$, an amplitude of oscillation, $s = U_\infty/\omega$, of 0·9 cm, $c = 1{\cdot}42\times10^5\,\mathrm{cm.sec}^{-1}$, and $\nu = 0{\cdot}0117\,\mathrm{cm}^2\mathrm{sec}^{-1}$, as in Schlichting's (1932) experiments, we obtain the following values for the basic parameters of § 2:

(i) $U_\infty/c = 2\times10^{-5}$.
(ii) $\omega d/c = 1{\cdot}7\times10^{-4}$.
(iii) $\alpha = 0{\cdot}11$.
(iv) $\beta = 0{\cdot}5\times10^{-4}$.
(v) $R_s = 250$.
(vi) Additionally we note that the bounding water tank had dimensions 340 cm by 55 cm, and was 44 cm deep filled with water. The fluid motion was rendered visible by particles of tin foil on the free water surface.

We may conclude from the values of the parameters (i) to (v) that the experiments fall within the purview of this theory, though $\alpha = 0{\cdot}11$ is rather large; but since, from (2.8), the modification to a given steady or harmonic component is of order α^2, even this value of α is reasonable. On the other hand the experimental conditions noted in (vi) suggest that the steady flow observed by Schlichting was affected both by the side walls and by the presence of a free surface. Qualitatively, however, the theory is in accordance with experiment, in that a steady flow is predicted to be directed away from the cylinder in the direction of oscillation.

5. Concluding remarks

The conditions under which the present theory is valid have been given in detail at the end of § 2 and considered in relation to a particular experiment in § 4. If U_∞, ω and d are characteristic velocity, frequency and length, and if c is the sound speed and ν is the kinematic viscosity, the following are *required to be small*:

$$\textit{Conditions (i) to (v)} \quad U_\infty/c, \quad \omega d/c, \quad U_\infty/\omega d, \quad \nu/\omega d^2, \quad \omega\nu/U_\infty^2. \tag{5.1}$$

In addition, *condition* (vi), we have assumed that other solid boundaries are at infinity. If there are such boundaries present, at some typical distance D, we may expect the effect on the steady streaming to occur through a parameter

$$(d^2/D^2)(\omega\nu/U_\infty^2), \tag{5.2}$$

which represents the ratio of the square of the thickness of the outer layer to D^2. If this parameter is large or of order 1 then the outer boundaries will be very important, but if it is small such boundaries will have little effect on the streaming.

In the form given in this paper, the double-layer theory of steady streaming has been applied to standing wave motions in incompressible flow. The application of similar ideas would be of great interest for compressible flow, as when sound waves are incident upon an obstacle, and for travelling wave motions, as when a water wave travels over a body of liquid with finite depth and rigid bottom (Hunt & Johns 1963; Longuet-Higgins 1953, 1960). In problems of the latter type it is usually important to include effects of bounding free or solid surfaces.

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Appendix 1

The neglect of the interaction between the potential flow and the steady flow in the outer layer

In the boundary-layer equation (2.2) we write

$$\psi = \psi_0(x,t) + zU(x,t) + \psi_a(x,z,t), \tag{A 1}$$

where $(\psi_0 + zU)$ is the periodic potential flow (including a displacement effect) and ψ_a is an additional flow, of which we are especially interested in the steady part. Since boundary-layer theory is assumed to be valid, the potential flow balances the given pressure gradient: thus (2.2) becomes

$$U\frac{\partial^2\psi_a}{\partial x\,\partial z} + \frac{\partial U}{\partial x}\frac{\partial \psi_a}{\partial z} - \left(\frac{\partial\psi_0}{\partial x} + \frac{z\,\partial U}{\partial x}\right)\frac{\partial^2\psi_a}{\partial z^2} + \frac{\partial^2\psi_a}{\partial t\,\partial z} + \frac{\partial\psi_a}{\partial z}\frac{\partial^2\psi_a}{\partial x\,\partial z} - \frac{\partial\psi_a}{\partial x}\frac{\partial^2\psi_a}{\partial z^2} = \nu\frac{\partial^3\psi_a}{\partial z^3}. \tag{A 2}$$

We average (A 2) with respect to time and denote the average of ψ_a by ψ_s, writing

$$\psi_a = \psi_s + \psi_t,$$

where ψ_t is the time-dependent part of ψ_a.

Then we have

$$J + \frac{\partial\psi_s}{\partial z}\frac{\partial^2\psi_s}{\partial x\,\partial z} - \frac{\partial\psi_s}{\partial x}\frac{\partial^2\psi_s}{\partial z^2} = \nu\frac{\partial^3\psi_s}{\partial z^3}, \tag{A 3}$$

$$J \equiv \overline{U\frac{\partial^2\psi_t}{\partial x\,\partial z}} + \overline{\frac{\partial U}{\partial x}\frac{\partial\psi_t}{\partial z}} - \overline{\left(\frac{\partial\psi_0}{\partial x} + z\frac{\partial U}{\partial x}\right)\frac{\partial^2\psi_t}{\partial z^2}} + \overline{\frac{\partial\psi_t}{\partial z}\frac{\partial^2\psi_t}{\partial x\,\partial z} - \frac{\partial\psi_t}{\partial x}\frac{\partial^2\psi_t}{\partial z^2}}, \tag{A 4}$$

where an overbar denotes an average with respect to time.

In the problem discussed in this paper ψ_s, being generated by an 'imposed' tangential velocity condition at the inner edge of the outer layer, is taken to be the dominant part of ψ_a. With U of order U_∞ and $\partial\psi_s/\partial z$ of order $U_\infty^2/\omega d$, as given by (2.12), an expansion suggests that $\partial\psi_t/\partial z$ has order $U_\infty^3/\omega^2 d^2$. Consequently, typical terms of J may be of the same order as the explicit non-linear terms in (A 3). We assume, however, that any part of ψ_t, with the same phase as ψ_0 or U, only yields terms in J of smaller order than the explicit (self-interaction) non-linear terms of (A 3). Typically this implies the assumption

$$\overline{U\frac{\partial^2\psi}{\partial x\,\partial z}} \ll \frac{\partial\psi_s}{\partial z}\frac{\partial^2\psi_s}{\partial x\,\partial z}. \tag{A 5}$$

With this crucial (non-interaction) hypothesis we have

$$\frac{\partial \psi_s}{\partial z}\frac{\partial^2 \psi_s}{\partial x\,\partial z}-\frac{\partial \psi_s}{\partial x}\frac{\partial^2 \psi_s}{\partial z^2}=\nu\frac{\partial^3 \psi_s}{\partial z^3}. \tag{A 6}$$

In fact Riley (1965) has now shown by an 'inner-outer' expansion, which gives the forms of ψ_s and ψ_t, that $J \equiv 0$ to first order because $U(x,t)$ and the dominant part of ψ_t differ in phase by $\frac{1}{2}\pi$; Riley has thus given final mathematical justification to (A 6) as the governing equation for the dominant part of ψ_s.

Appendix 2

Similarity solutions, together with a discussion of the problem of a torsionally-oscillating disk

We note those cases in which (2.16), subject to (2.17), reduces to an ordinary differential equation. To this end we write

$$\phi = k(\xi)\,X(\xi)\,g(\lambda), \quad \lambda = \zeta/k(\xi), \tag{A 7}$$

and obtain

$$\left.\begin{aligned} g''' + \beta g g'' - \alpha g'^2 &= 0,\\ \beta = k(kX)', \quad \alpha &= k^2X', \end{aligned}\right\} \tag{A 8}$$

where a prime denotes an appropriate derivative and α and β are constants. An exhaustive discussion of (A 8), on the lines of that given by Jones & Watson (1963, § V, 21), would be inappropriate here, but we note particularly the following possibilities:

(i) $$X = A(\xi-\xi_0)^{\alpha}, \quad k = A^{-\frac{1}{2}}(\xi-\xi_0)^{\frac{1}{2}(1-\alpha)}, \tag{A 9}$$

where A, α and ξ_0 are constants, with

$$g''' + \tfrac{1}{2}(1+\alpha)\,gg'' - \alpha g'^2 = 0; \tag{A 10}$$

(ii) $$X = A\,e^{c\xi}, \quad k = A^{-\frac{1}{2}}\,e^{-\frac{1}{2}c\xi}, \tag{A 11}$$

where A and c are constants, with

$$g''' + c(\tfrac{1}{2}gg'' - g'^2) = 0. \tag{A 12}$$

Equation (A 12) is a limiting form, as $\alpha \to \infty$, of (A 10).

The boundary conditions for (A 8), (A 10) and (A 12) are

$$g'(0) = 1, \quad g(0) = 0, \quad g'(\infty) = 0. \tag{A 13}$$

As a matter of interest, it may be noted that Mr J. Watson has given simple proofs that (A 10) and (A 12), subject to (A 13) have no solutions for $\alpha \leqslant -1$ and $c \leqslant 0$.

A simple solution of (A 10) and (A 13) for $\alpha = 1$ is

$$g = 1 - e^{-\lambda}, \tag{A 14}$$

which is equivalent to (3.19). Another simple solution occurs for $\alpha = -\frac{1}{3}$, for which (A 10) and (A 13) yield

$$g = 6^{\frac{1}{2}}(1-e^{-\lambda\sqrt{(\frac{2}{3})}})/(1+e^{-\lambda\sqrt{(\frac{2}{3})}}). \tag{A 15}$$

Solutions for other values of α or c may be obtained by the method of § 3, with an expansion like (3·2): but γ is a constant and ϕ_n is a function of λ only. A relation analogous to (3.16) then determines γ and completes the solution. The method is used on another problem in the next paragraph.

A similarity solution of a corresponding three-dimensional problem has already been discussed by Rosenblat (1959), in connexion with the torsional oscillations of an infinite disk. The similarity there is of a different kind, the radial and azimuthal velocity components each being proportional to a product of the radius and a function of the distance normal to the disk. However, the analysis leads to a differential equation for the outer layer of the form (A 8), and it is conveniently discussed here. In Rosenblat's paper a function $f(\eta)$ is given by his equation (57): but with the exponential term neglected and when subjected to a simple transformation $\eta = \zeta\sqrt{2/\epsilon}$, $f = q\sqrt{2/4\epsilon}$, that equation yields the following equation for the outer layer

$$q''' + 2qq'' - q'^2 = 0, \tag{A 16}$$

$$q = 0, \quad q' = 1, \quad \zeta = 0; \quad q' \to 0 \quad \text{as} \quad \zeta \to \infty. \tag{A 17}$$

The Fettis method yields

$$q = \gamma - \frac{1}{2\gamma}e^{-2\gamma\zeta} + \frac{1}{16\gamma^3}[-e^{-2\gamma\zeta} + \tfrac{1}{2}e^{-4\gamma\zeta}] + \frac{1}{128\gamma^5}[-e^{-2\gamma\zeta} + e^{-4\gamma\zeta} - \tfrac{1}{3}e^{-6\gamma\zeta}] + \ldots, \tag{A18}$$

where γ must be chosen to satisfy $q(0) = 0$.

If terms $O(\gamma^{-1})$ are retained we obtain $\gamma = 0{\cdot}707$; to $O(\gamma^{-3})$ we have $\gamma = 0{\cdot}746$; and to $O(\gamma^{-5})$ we have $\gamma = 0{\cdot}751$. This value agrees with Benney's (1964) calculation.